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EDITED BY

EARLE R. HEDRICK
DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

HAROLD T. DAVIS HARRY W. KUHN
THEOPHIL H. HILDEBRANDT DERRICK N. LEHMER
JOHN R. KLINE HENRY L. RIETZ
CAROLINE E. SEELY

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Florian Cajori

February 28, 1859–August 14, 1930

FLORIAN CAJORI

On August 14, 1930, at his home in Berkeley, California, Professor Florian Cajori, one of the most prolific writers on the history of mathematics, physics, and astronomy in this generation, died in the seventy-second year of his age. For some time his health had been failing, and being further weakened by a hospital operation his system was unable to withstand the attack of pneumonia which proved fatal.

He was born at St. Aignan, near Thusis (Canton of Grisons), Switzerland, on February 28, 1859, and came to the United States in 1875, at the age of sixteen. His early education was received at Zillis (Thusis) and Chur, Switzerland. Soon after coming to America he entered the State Normal School at Whitewater, Wis., where he remained for two years (1876–1878). A little later he entered the University of Wisconsin, graduating with the degree of B.S. in 1883, and receiving the degree of M.S. in 1886. After taking the bachelor's degree he spent a year and a half at Johns Hopkins University (Jan., 1884–June, 1885). He was then elected assistant professor of mathematics (1885) at Tulane University, becoming professor of applied mathematics two years later (1887). In 1889 he went to Colorado College as professor of physics, which position he held for nine years. From 1898 to 1918 he was professor of mathematics in the same institution, serving also as dean of the department of engineering from 1903 to 1918. In 1918 he was called to the University of California to accept the position of professor of the history of mathematics, doubtless the first man ever to receive this title, and one of the few to have a university position allowing the entire time of its occupant to be given to research, writing, and teaching in this particular field. The appointment, which dated from July 1, 1918, was an exceptional honor and the result amply justified the university in affording the unique opportunity. On July 1, 1929, he retired from active teaching, with the title of professor emeritus.

Professor Cajori's literary activity in his chosen field was far more extended than is generally recognized. The list of his articles in scientific journals is extensive, and the subjects treated give evidence of his wide range of interests in the field of the history of science.

Among the publications to which he was a contributor may be mentioned the following: American Journal of Mathematics, American Mathematical Monthly, Annals of Mathematics, Archeion, Archivio di Storia della Scienza, Bibliotheca Mathematica, Bulletin of the American Mathematical Society, L'Enseignement Mathématique, Isis, Silliman's Journal, Loria's Bollettino, Mathematical Gazette (England), Nature (London), Mathematics Teacher, Popular Astronomy, Popular Science Monthly, Revista Matematica Hispano-Americana, School and Society, School Science and Mathematics, Science, Scientia, Scientific Monthly, Schlömilch's Zeitschrift, and the serial publications of both the University of California and Colorado College. To these periodicals he contributed no less than 140 articles (including reviews) during the period of his residence at the University of California,—an average of nearly one each month. The total number of items in his bibliography during

his forty years of active writing was, however, much greater, being between three and four hundred. It is expected that this feature of his work will be treated *in extenso* by Professor Archibald in an article to appear in Isis. Mention should also be made of his textbooks.

He will, however, be best known by his numerous historical works, the publication of which extended over a period of forty years. The first of these works was *The Teaching and History of Mathematics in the United States* (Washington, 1890). Although quite naturally lacking in the scholarship and thoroughness displayed in the publications of the last years of his life, it laid out a line of study which was quite new in this country and which served a worthy purpose in creating an interest in the subject.

His next three works were in the nature of general histories in his fields of major interest. The *History of Mathematics* appeared in 1894, the *History of Elementary Mathematics* in 1896, and the *History of Physics* in 1899. Each was written under conditions that were evidently not favorable to exhaustive research in the original sources from which history should draw; indeed, even the secondary sources of information were too limited to admit of the work of which he afterward showed himself capable. His first opportunity to display his real ability came when Moritz Cantor invited him to contribute a section on arithmetic to volume IV of the *Vorlesungen über Geschichte der Mathematik* (Leipzig, 1908). This was a distinct honor, and the rule of *noblesse oblige* showed itself. He went to the sources and was thus enabled to write as one having authority.

Thus there opened a new era in his work,—the era in which he found himself. In the next year there appeared his *History of the Logarithmic Slide Rule* (1909), involving a line of investigation apparently suggested to him partly by his duties as dean of the department of engineering, and partly by his studies in connection with the Cantor project. He was no longer content with depending upon the works of others; others could now depend upon him. It would seem that it was the study of the slide rule that led him to write his first biographical work, the *William Oughtred* (1916), in which he gave a worthy résumé of the life of "a great seventeenth-century teacher of mathematics" (using the word "teacher" in a broad sense) and, indeed, of one who may perhaps justly be called the greatest in England. At about this time (1917 and 1919) he gave a much-needed revision to his two histories of mathematics, improving both their style and content. The *History of Elementary Mathematics* was translated into Japanese in 1928, and the *History of Physics* was revised in 1929. His other venture in the field of biography was made much later, after he had gone to Berkeley,—*The Chequered Career of Ferdinand Rudolph Hassler, first Superintendent of the United States Coast Survey* (1929). In this he gave evidence of extensive search after material and of having found it and skillfully used it to make this description of the life and works of his compatriot a worthy contribution to the history of science in their adopted country.

Three special studies in the history of mathematics stand out, however, above all the rest, and by these he will longest be remembered. They illustrate the words of the greatest of the poets laureate of England,

> "That men may rise on stepping stones
> Of their dead selves to higher things;"

for, after all, Professor Cajori's earlier works were as mere stepping stones to those of his last few years at Berkeley. The first was his *History of the Concepts of Limits and Fluxions in Great Britain from Newton to Woodhouse* (1919), which may be looked upon as a kind of inaugural address at the University of California. It gave evidence of a more careful examination of sources than any of his previous works, even the ones on the slide rule and on Oughtred. It was in the preparation of this work, no doubt, that he found a stimulus for his essay on *Newton's Fluxions* and the one on the Law of Gravitation, each of which he prepared for the Bicentenary meeting in honor of Newton, held at New York in 1927, and perhaps for the edition of the *Principia* and *Optics* upon which he had nearly completed the work at the time of his death, and which will, it is understood, be published.

The second of his special studies was *The Early Mathematical Sciences in North and South America* (1928), a series of essays rather than a connected discourse, but none the less a genuine contribution to the history of science in the New World. It is no disparagement of the work to say that its value lies no less in the extensive bibliographies at the close of the several chapters than in the text itself, for not only do these show a wide range of reading on the part of the author but they will prove of great value to future students in this field. Naturally the work deals with astronomy, geodesy, and physics rather more than with pure mathematics,—the result of the early influence of England and Spain at a time when these three subjects were uppermost in the applications of the science. In the same year (1928) he published his *Mathematics in a Liberal Education*, an appeal for a better knowledge of the purpose of mathematical training.

The third of the special studies will, however, stand out as the real monument to Professor Cajori's life work, *The History of Mathematical Notations* (2 vols., 1928, 1929). It is in this that he appears at his best, and long after our current histories of mathematics are forgotten, this will remain as one of the most important contributions of our time to any of the special phases of the general subject. That he lived to see the conclusion of this work, and to see the worthy form in which it came from the press, and also to see the practical conclusion of his manuscripts on the *Principia* and the *Optics*, must have been a source of special gratification to him as it is to scholars generally throughout the world of science.

Of the honors conferred upon him it is not necessary to speak at any length. Gratifying as they were, they weigh but little in comparison with his accomplishments. He held the degrees of Ph.D. (Tulane, 1894), LL.D. (University of California, 1912, and Colorado College, 1913), and Sc.D. (Wisconsin, 1913). He was a member of the American Mathematical Society, the Mathematical Association of America (president, 1917), the Deutsche Mathematiker-Vereinigung, the American Association for the Advancement of Science (vice president of Section L, 1923), the Mathematical Association (England), The National Council of Teachers of Mathematics, the History of Science Society (council member), and the Circolo Matematico di Palermo, and was a fellow of the American Academy of Arts and Sciences.

On September 3, 1890, at St. Louis, Mo., he was married to Elizabeth G.

Edwards, who survived him, as did their only son, Lieut. Florian Anton Cajori, who served in France during the World War.

It is impossible to summarize in a few words the esteem in which Professor Cajori was held by those who knew him either personally or through his writings. His gentle manner and unfailing kindness endeared him to all with whom he was associated, and at the same time there was a bond between him and many thousands whom he led to appreciate more fully that range of knowledge which makes up what a Hindu master so appropriately called "the science venerable."

DAVID EUGENE SMITH

LEVI-CIVITA ON MECHANICS

Lezioni di Meccanica Razionale. By Tullio Levi-Civita and Ugo Amaldi. Vol. 1, 2d Edition, Vol. 2, parts 1 and 2, 1st Edition. Bologna, Nicola Zanichelli, 1926–30. Pp. 807, 526, 684.

These books take their place along with such treatises as those of Routh, Appell, Budde, Resal, and Lecornu, both in quality and extensiveness. Together they comprise a good many pages. Nevertheless they are limited to the statics and dynamics of particles and rigid bodies. They do not digress extensively into special fields; no matter is set off in small type to invite omission; the principles and methods of mechanics are developed from the more elementary to the more advanced; the applications given are those of which the importance or interest is readily seen. There is no effort at terseness or brevity of expression. The authors are not only able, but are willing to lay the matter fully before the reader. The work is not merely a sequence of mathematical formulas pieced together with a minimum of sentences, whose meaning may, or may not, be fully comprehended by the reader. Equations in the last analysis must be the supporting elements of any work on dynamics. But once the equations have been definitely discovered and accepted, little skill may be required to piece them together in something like a logical sequence. But a person who does this is rather a compiler than an author. The present volumes are a real narrative, interesting and readable, worthy of occupying a high place in a literature already distinguished.

The work opens with a chapter on vector analysis. No work on mechanics can be regarded as fully adequate that shuns the use of vectors where they undoubtedly contribute to clarity and simplicity. In the present case they are used judiciously, appearing here and there throughout the entire work. The introductory chapter develops the subject distinctly as a tool, and one is not led astray through a mere attachment for the subject.

The next five chapters, comprising a total of nearly 250 pages, are devoted to kinematics. They state conspicuously at the start that mechanics presupposes geometry, and adds to this, as its own peculiar fundamental concept the notion of time. Attention is called to the fact that within recent years a great change has taken place in our ideas of time, which formerly was taken as primitive and dismissed at that. Only a page is devoted to the newer relativity aspect of time, but the cardinal idea is set forth, and enough is said perhaps to arouse such interest in the reader as will lead him to consult fuller and special treatments. The plan of developing at the beginning all matters of a kinematical nature, whether they pertain to a particle or a rigid body, has of course a logical advantage. But it has the disadvantage of delaying the real subject, that of dynamics. It is not until more than 300 pages have been read that one encounters the idea of a force; differential equations of motion first occur on page 358. Some of the kinematical ideas that relate to a rigid body are not simple. It is likely that even an attentive reader will find it necessary, when he finally reaches the dynamics of the rigid body, to study again such a question as that of the angles of Euler, quite as if it were all new material.

One is pleased to find the consideration of dynamics proper starting with a chapter bearing such a title as "The Concepts and Fundamental Postulates of Mechanics." One reads with attentive interest the way in which the idea of force is developed, and following it, that of mass. A careful study of the forty pages of the chapter should answer some of the questions that often trouble the careful student.

Volume II, part 1, treats the more customary problems of particles and systems. The elements of celestial mechanics receive a fairly satisfactory chapter of 50 pages, including a few words on perturbations. The equations of Lagrange are introduced, and one encounters the symbols of Christoffel. The concluding chapter, of considerable length, is devoted to stability and vibrations.

The second part of volume II opens with the systematic study of the rigid body, and the matter is carried through about 300 pages. Then comes the portion of the work that one looks through with particular interest—the discussion of the more advanced aspects of dynamics. In such questions as the derivation and discussion of the canonical equations, the avoidance of an extreme terseness is particularly welcome. The transformation theory of dynamics, of which there are not many clear and satisfactory expositions, can be read here with understanding. The Hamilton-Jacobi theory is well set forth, and the splendid astronomical application is carried farther than in most works. But the perturbation theory of astronomy is such an extensive subject, so full of peculiar difficulties, that one can doubt if mere hinting at it can give the reader any adequate conception of it at all. The work closes with a chapter of nearly a hundred pages on impulsive motion.

A very notable feature of the work is the biographical notes. Whenever any mathematician is mentioned, and care is taken to assign the more important parts of the subject, a note occurs that sets forth the main facts about the life and work of the man. These notes sometimes glow with a warmth of appreciation that cannot fail to impress the reader.

Each chapter is followed by a list of exercises. They are of a very substantial character; at times they constitute important supplements to the text. In one instance the exercises cover 35 pages, in another 32 pages. In all, the exercises comprise 300 pages out of a total of 2000. The authors have not only produced a text proper of unusual clearness; they have given an abundance of material for the exercise and development of the reader's own ability.

K. P. WILLIAMS

FISHER ON INTEREST

The Theory of Interest as Determined by Impatience to Spend Income and Opportunity to Invest it. By Irving Fisher. New York, The Macmillan Company, 1930. xxvii+566 pp.

This book was begun as a revision of the author's *The Rate of Interest* (1907) which has long been out of print. In the *Preface*, Professor Fisher remarks that his theory of interest has scarcely been altered, but its exposition has been so amplified and recast that it may seem more changed to those who misunderstood his first book than it does to him. The result of this thorough revision has been a new book, a complete rewriting of the former book with additions of new material.

Although scarcely what a mathematician would call a book on applied mathematics, this work contains numerous applications of algebraical and graphical methods and has a distinct mathematical tone throughout. A mathematician reading the book will soon become aware of the fact that the author is devoting the first ten chapters to a description of a mathematical model. This model he reproduces in the next four chapters. Such a reader may think that Professor Fisher has chosen a long way around, but he is writing to a larger audience. On p. 312 he remarks "if I were writing primarily for mathematically trained readers, I would have reversed the order, giving the first place to the formulas, following these with the charts for visualization purposes and ending with verbal discussion."

Professor Fisher's theory of interest is, as he states, p. ix, an enlargement and synthesis of already existing partial theories, and is based on *investment opportunity*, *human impatience* (preference for comparatively early income over comparatively remote or deferred income) and *market exchange* (buying and selling). The assumptions underlying his general theory are as follows.

I. *Investment Opportunity*, (a) each individual has a choice within limits of different income curves $y(t)$ where t is time, and to each curve is associated a risk; (b) the individual selects the income curve which produces a maximum present value, where the present value takes into account the risk element.

II. *Human Impatience*, (a) the degree of impatience of any individual depends upon his income stream, $y(t)$, as chosen by him and modified by exchange; (b) each person, after or while first choosing the option of greatest present value, will then modify it by exchange so as to convert it into that form most wanted by him.

III. *Market Exchange*, (a) the rate of interest must equalize supply and demand, and (b) the expected present value of all loans equals the present value of the borrowings, but due to risk there may be a wide discrepancy between the actual realization and the original expectation.

Hypotheses Ib and IIb are best handled by means of the calculus, but because of the unprepared state of most expected readers, the author relegates the derivation of the necessary conditions to an appendix. Incidentally it may be remarked here that in all previous somewhat similar books the author has consigned all mathematics to appendices.

In the first eight chapters and in the mathematical treatment (Chapters XI–XIV) Professor Fisher assumes that the probability of risk is zero. In Chapter XIV he remarks: "To attempt to formulate mathematically in any useful, complete manner the laws determining the rate of interest under the sway of chance would be like attempting to express completely the laws which determine the path of a projectile when affected by random gusts of wind." He does, however, give a qualitative discussion of what would happen if risk were taken into account. It would seem that by assuming a law of risk obtained from empirical data in a way similar to the way in which a mortality table is drawn up, and by writing the demand as a functional defined on a range extending from some time in the not too remote past to a time in the future as I have indicated in my paper in the Journal of Political Economy, Oct. 1927, one might formulate a mathematical theory of interest which would take risk into account. The question of whether or not such an extended theory would be worthwhile from a practical standpoint would of course remain open.

In the mathematical treatment the author assumes that all of the income for each year is concentrated at the middle of the year (p. 288). If he had not made this restrictive assumption, the problem would have become one in the maxima of functionals instead of one in the maxima of functions. One wishing to develop further the theory of interest from this point of view may find meager hints in a related theory in my papers in this Bulletin, March–April, 1928, and the American Journal of Mathematics, January, 1928.

C. F. Roos

TIMOSHENKO ON STRENGTH OF MATERIALS

Strength of Materials. By S. Timoshenko. Two volumes. New York, Van Nostrand Company, 1930. xii+368 pp; xii+401–735 pp.

The recent appearance of these texts on elasticity will be welcomed by engineering colleges and technical schools for many reasons. Perhaps foremost is the fact that it is written in English and fills a large gap between the two extremes of required courses of strength of materials in our engineering colleges and the comprehensive treatise by A. E. H. Love, *The Mathematical Theory of Elasticity*, now in its fourth edition.

Today it is becoming increasingly more difficult for the designer to meet the needs of greater working stresses, as required by more rapidly moving machine elements, unless analytical methods replace "rules of thumb." To further this end the author has provided a mine of interesting methods both for the student in the classroom and the research man in the laboratory. To quote the author's preface: . . . . "At the present time a decided change is taking place in the attitude of designers towards the application of analytical methods in the solution of engineering problems . . . . . The importance of analytical methods combined with laboratory experiments in the solution of technical problems is becoming generally accepted . . . . . It is the aim of this book (vol. I) to present problems such that the student's attention will be

focussed on the practical applications of the subject . . . . . The second volume is written principally for advanced students, research engineers and designers."

The fact that industries are making it more urgent that engineering students have a thorough basic training, is finding a response by our colleges in that some of them are organizing for a larger program by creating a separate department of applied mechanics, so styled or labelled by some other suitable name. In this connection Hovgaard (Science, vol. 71, No. 1840, p. 347) writes concerning theoretical mechanics (which includes mechanics of elasticity): ". . . . On the whole it appears that in the United States this subject is too often taught in a scattered way, and as a subsidiary science, of which each department gives only what is required for its spe ific needs without much regard to unification of the science . . . . . It is time that theoretical mechanics be given its appropriate place in the curriculum of American engineering schools. As a means of training in scientific methods of thinking and as a mental discipline introductory to research work, the subject is probably unequaled." To promote these ends, it is the opinion of the reviewer that the author has made a large contribution.

The chapter headings for Part I are: Tension and Compression Within the Elastic Limit, Combined Stress, Torsion, Stresses in Beams, Deflection of Beams, Statically Indeterminate Problems of Bending, Plastic and Non-Linear Bending, Stresses Due to Direct and Bending Loads, Combined Bending and Twist, and Energy of Strain; Part II: More Complicated Problems in Bending of Beams, Curved Bars, Thin Plates, Deformations in Bodies Generated by Revolution, Buckling of Bars, Plates, and Shells, Stress Concentration, and Mechanical Properties of Materials.

Part I is based upon the material usually covered in engineering courses, although it is more comprehensive in the types of applications. No apology is made for abundant use of differential and integral methods. As early as p. 19 appears the expression

$$\left(1+\frac{x}{n}\right)^{n}_{n=\infty}=e^{x}$$

which is intended for $\lim_{n\to\infty}(1+x/n)^n$. This limit is used to obtain a well known specific result from a general case, a type of analysis frequently used by the author. Part I is characterized by many special features such as Mohr's circle for combined stresses, conjugate beams for obtaining deflection, principle of superposition for statically indeterminate systems, tangent property of ellipse of inertia for determining neutral axis when the plane of deflection is not the plane of the bending forces, and use of integral methods before the method of differential equations. It is reserved for the final chapter to show the great utility of energy methods, particularly the application of Castigliano's Theorem and the Theorem of Least Work to the solutions of statically indeterminate problems, also the Reciprocal Theorem of Betti for obtaining direct results, especially the so-called "line of influence."

The reviewer had the opportunity of completing a study of Part I before the second volume was off the press. In view of the increasing scope as indicated by frequent references in Part I to more extended theory in the second volume, it was somewhat disappointing not to find a chapter on general equa-

tions of equilibrium. One might have anticipated the biharmonic equation for the stress function or for deflection of plates (the second-order equation in curvature is used for the circular plate in Chapter 3). An excellent treatment of minimizing the potential energy to derive equilibrium equations is given in Chapter 2 of the *Handbuch der Physik* (1928), a monumental treatise to which no reference is made. Pursuing the author's purpose . . . . "to prepare a book which contains new developments that are of practical importance . . . . without going beyond the limit of the usual standard in engineering mathematics . . . . ," he presents practical applications of trigonometric series to problems of beams, energy methods to curved bars and stability problems, membrane and electric analogs to stress concentration at reentrant corners of rolled sections, and a description of the photoelastic method for investigating stresses near filets and holes in tension and compression members. The concluding chapter on materials and strength theories is non-mathematical.

The typography and paging is excellent. (The lapse in paging between the two parts is to allow revision without changing the folios.) Nearly five hundred well executed figures aid in the visualization so necessary to the student in this analysis. The table of notations given at the beginning of each book largely follows the German system. About four hundred examples are given of which three-fifths are accompanied by suggestions for their solution. Many of these doubtless have come from the author's wealth of experience.

The books are well indexed both as to contents and to authors, listing some three hundred writers with references to over five hundred cited papers. Only about forty per cent of the citations are in English. This seems to be adequate indication that the foreign language requirement should be stressed, particularly for graduate students. Modesty prevents the author from including his name in the index of authors, although footnotes refer to some thirty of his published articles.

D. L. HOLL

SHORTER NOTICES

Grundlagen der Hydromechanik. By L. Lichtenstein. Berlin, Springer, 1929. xii+506 pp.

This book by the well known mathematician of the University of Leipzig constitutes volume 30 of the Springer collection, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. As might be expected, it is very different from every other treatise on hydrodynamics. For the author, the fundamental problem of hydrodynamics is the integration of certain systems of partial differential equations with assigned boundary conditions, and he calls to his aid all the resources of modern mathematics. This makes necessary a large amount of preliminary material; the first chapter is devoted to topology, the second gives a brief but interesting account of vector analysis, the third treats potential theory, a subject to which the author has made valuable contributions, and so on. It is not until we reach page 290 that the equations of motion are derived. One of the most interesting and important chapters of the book is that devoted to the propagation of discontinuities, a difficult subject, the understanding of which is essential to any comprehension of wave motion. The last chapter is devoted to existence-theory questions.

The book could not profitably be recommended to a beginner, but to one who has studied Lamb and who wishes to understand more fully the mathematical foundations of the subject it should prove very valuable. For practical applications of the theory one must still turn to Lamb and to some such book as Tietjens' account of Prandtl's lectures on hydro- and aero-mechanics. The printing has the degree of excellence we have come to expect in the books of the Springer collection.

F. D. MURNAGHAN

Les Méthodes Nouvelles en Analyse Quantique. By Julien Pacotte. Paris, Blanchard, 1929. viii+139 pp.

The rapid development of the newer quantum theories of atomic structure during the past five years with the differing mathematical methods of the matrix mechanics and the wave mechanics has rendered particularly necessary and desirable the attempt to examine critically the various treatments, correlating them and ascertaining their similarities and differences, so that ultimately a synthesis shall result which may justly be called *the* quantum mechanics. It is a preliminary critical study toward this end which is provided by the present volume, which thus lays no claim to originality in its material content but provides a brief digest of most of the principal work in quantum mechanics through the year 1928.

The author begins with a rapid résumé of the principles of the Bohr theory —the quantum conditions, correspondence principle, etc. At the very outset he inserts comparisons between this early theory and the newer points of view of Heisenberg, de Broglie and Schrödinger. These are rather illuminating to the reader who already has at his command a reasonably thorough knowledge

of these theories, but will probably carry little if any meaning to the person approaching the subject for the first time.

In the second chapter the author presents a condensed summary of the matrix mechanics of Heisenberg, Born, and Jordan. The treatment is sketchy, purely analytical and stated in general terms with no illustrative problems. In a later chapter the linear oscillator and the hydrogen atom *are* worked out comparatively from both matrix and wave equation points of view. These are the only illustrative applications in the book. In the second chapter mention is also made of Dirac's q-number theory and Schrödinger's operators and their corresponding matrices.

Chapter III is devoted to the de Broglie waves, where the treatment, due to the compression of the material, lacks the clarity and grace of de Broglie's own papers and monographs. This is followed by chapters on Schrödinger's wave mechanics, the perturbation theory in quantum mechanics, wave mechanics and relativity (in which the five-dimensional theory of Kaluza and Klein is discussed), and finally the general transformation theory with its statistical implications. A few pages are here devoted also to the new statistics of Bose-Einstein and Fermi-Dirac.

The reader who is familiar with the details of the current theories will find much that is stimulating in this book, even though it can hardly be termed a thoroughgoing critique.

R. B. LINDSAY

An Introduction to the Geometry of n Dimensions. By D. M. Y. Sommerville. New York, E. P. Dutton, 1930. 196 pp.

In recent years books on some phase of n-dimensional geometry have appeared in all languages. In England and America, however, there seemed to be little interest in this subject before the appearance of general relativity, and the interest then was connected with Einstein. In the preface Sommerville tells us that Englishmen were among the first to write on this subject, and that the subject was entirely neglected until recently; now, however, there are signs of a revival of interest. This book is a most valuable addition to the English literature of the subject.

The aim of the author is not to write an introduction to the Einstein theory but rather to select topics which will reveal to the reader the inherent beauties and surprises of the subject. Neither has he confined himself to metric, projective, or euclidean geometry, but has used freely the ideas of all three. The book starts with the fundamental concepts of incidence, parallelism, and perpendicularity (largely synthetic). Then follows the analytic treatment in which algebraic varieties are discussed, especially the quadric. We also find Plücker coordinates introduced and neatly applied. The applications of integral calculus are also given.

The last half of the book is devoted to the study of the polytope (analogue of the polyhedron) which is treated in considerable detail. This part of the book the reviewer found most fascinating both on account of the material chosen and the elegance of the treatment. The last chapter discusses the regular polytopes.

I would not call this an elementary book and yet I believe it could profitably be placed in the hands of the young student of geometry very early in his career.

C. L. E. MOORE

Principes Géométriques d'Analyse. By Gaston Julia. Paris, Gauthier-Villars, 1930. vi+116 pp., 35 figures.

The present volume has for its purpose the development and explanation of those geometric concepts which are employed in connection with rational, and particularly linear, transformations of a complex variable z, and the consequent transformations of uniform and of multiform functions of z.

The first chapter is concerned with circles of convergence, and is then extended to infinite regions by means of spherical stereographic projection and circular inversion, with special reference to domains including invariant points. The second chapter contains a systematic geometry of the circle, including a number of interesting particular cases. To make the treatment uniform a number of pages are employed to interpret the results in terms of non-euclidean (hyperbolic) geometry. In the third chapter these ideas are applied to (l, m) correspondences. The treatment here seems to the reviewer rather too restricted; in the preceding discussion free use was made of projectivity, of fixed points—distinct or coincident—but in this one no corresponding use is made of the involutions, branch points, coincidences and neutral elements that play such an extensive role in algebraic geometry. It is true that the geometry of the circle furnishes only a limited interpretation of these ideas, but the extension to the projective field would add considerably to its interest without entailing any digression greater than that employed in the preceding chapter.

Applications of the principles already established are made to solve a large number of problems in mapping. Free use is made of the lemma of Schwartz, which the author states in the form: Given a function $f(z)$ holomorphic within the unit circle and vanishing at the origin and with modulus less than unity, then $|f(z)| \leqq |z|$. If for $|z| = 1$ the equality sign holds, then $f(z) = e^{i\theta}z$.

The last chapter is devoted to the theory of the modulus and to the mean, both linear and of order p, of moduli of analytic functions, extended to multiply connected regions. This part is to be further extended in a later volume.

VIRGIL SNYDER

Cours de Géométrie, Pure et Appliquée. By M. d'Ocagne. Paris, Gauthier-Villars, 1930. vi+429 pp., 180 figures.

This work is a printed revision of the author's earlier mimeographed course in two volumes, designed primarily for students of the technical schools of France. It is introduced by a very rapid review of projective geometry of one to three dimensions, mostly from the analytic standpoint. This is followed by some fifty pages on differential geometry of curves and surfaces; a large number of properties are derived, almost all from the standpoint of infinitesimals, as employed by Bertrand in his treatise, but with many proofs replaced by simpler and more direct ones supplied by the author or by students who have taken the course in recent years. This part contains the essence of

the entire course. Although no differential equations are derived or employed, the characteristic features of lines of curvature, asymptotic lines, conjugate systems and geodesics are derived and applied to various surfaces, including a detailed study of developable and other ruled surfaces. A fairly full chapter on line geometry, both algebraic and differential, and one on the geometry of motion complete the part of the treatise concerned with pure geometry. References are frequent, but too many of them refer to alternate proofs provided by l'École Polytechnique, a large part of the original ones not being mentioned at all.

The second part includes the general theory of machines, a comprehensive treatment of roulettes and glissettes, general composition of motions and the general geometry of motion, with application to graphical kinematics, graphical statics, mechanical integration, description of various integrating machines and of the theory of graphical integration. This part, comprising about half of the volume, is well done; each step is explained in detail, with a discussion of mechanical limitations as well as of the geometric theory.

The book closes with a very condensed account of the theory of nomography, which is less successful. The author has attempted to reproduce the substance of his well known treatise on the subject in about forty pages.

For the general purpose of providing technical students with the theory and use of differential geometry in two and three dimensions, the book is a very valuable one.

VIRGIL SNYDER

NOTES

In addition to those of the editorial staff of the Bulletin, the following persons have assisted the editors, either by refereeing papers or by advising concerning papers offered for publication in the present volume: A. A. Albert, W. L. Ayres, E. T. Bell, H. F. Blichfeldt, R. D. Carmichael, A. B. Coble, P. H. Daus, L. E. Dickson, L. L. Dines, E. L. Dodd, Tomlinson Fort, T. C. Fry, W. L. Hart, T. R. Hollcroft, E. V. Huntington, W. A. Hurwitz, A. J. Kempner, W. A. Manning, A. D. Michal, C. N. Moore, Virgil Snyder, E. B. Stouffer, D. J. Struik, J. D. Tamarkin, H. S. Vandiver, G. E. Wahlin, Warren Weaver, J. H. M. Wedderburn, G. T. Whyburn, W. M. Whyburn, Norbert Wiener, R. L. Wilder, W. A. Wilson, B. C. Wong. The editors desire publicly to recognize this service.

The British Association for the Advancement of Science will meet in London in September, 1931. This will be the Association's centenary meeting.

The Planck gold medal has been awarded to Professor Niels Bohr, of the University of Copenhagen.

The Königsberger Gelehrte Gesellschaft announces the following subject for its prize to be awarded in January, 1932: *The discussion, by means of both function-theoretic and geometric (in particular, topological) methods, of the topological nature of the configurations formed by n analytic functions of n complex variables.* Competing memoirs should be sent to that society before October 15, 1931.

Professor Werner Heisenberg, of the University of Leipzig, has been elected a member of the Saxon Academy of Sciences in the section of mathematical physics.

Professor Arthur Korn, of the Charlottenburg Technical School, has received the honorary degree of doctor of engineering from the Breslau Technical School.

Charles Maurain, dean of the Faculty of Sciences of the University of Paris, has been elected member of the Paris Academy of Sciences in the section of astronomy, as successor to the late H. Andoyer.

Professor Max Planck has been elected president of the Kaiser-Wilhelm Gesellschaft zur Förderung der Wissenschaften.

Professor Arnold Sommerfeld, of the University of Munich, has been elected a foreign member of the Hungarian Academy of Sciences.

The University of Paris has conferred an honorary doctorate on Professor P. Zeeman, of the University of Amsterdam.

The University of Dublin has conferred an honorary doctorate on Mr. Edward Tyler, for his investigations in aerodynamics.

Princeton University has conferred an honorary doctorate on President K. T. Compton, of the Massachusetts Institute of Technology.

The following have been appointed National Research Fellows in mathematics for the academic year 1930–31 (this list includes reappointments, and may not be complete): R. P. Agnew, A. C. Berry, Leonard Carlitz, L. W.

Cohen, C. C. Craig, H. T. Engstrom, D. H. Lehmer, S. B. Littauer, N. H. McCoy, E. J. McShane, Gordon Pall, W. T. Reid, N. E. Rutt, W. J. Trjitzinsky, Jacob Yerushalmy.

Mr. O. J. Farrell has been appointed Rockefeller International Research fellow in mathematics.

Under the will of the Norwegian statesman Christian Michelson, a Research Institute has been created at Bergen, Norway, to which scientists may be called for periods of five years. Professor T. Skolem, of the University of Oslo, has been designated for the mathematical sciences.

Dr. Hans Falkenhagen, of the University of Cologne, has been promoted to an associate professorship of theoretical physics.

Professor H. Fehr, of the University of Geneva, editor of L'Enseignement Mathématique, has been appointed rector of the University.

Dr. S. Mandelbrojt, of the University of Clermont, has been promoted to a professorship of rational and applied mechanics.

Professor Adalbert Prey, of the German University of Prag, has been appointed professor of theoretical physics at the University of Vienna.

Professor Hermann Weyl, of the Zurich Technical School, has been appointed professor of mathematics at the University of Göttingen, as successor to Professor David Hilbert, retired.

Dr. Heinrich Wieleitner has been appointed to an honorary professorship at the University of Munich.

Mr. Alexander Oppenheim has been appointed lecturer in mathematics at the University of Edinburgh.

Mr. J. G. Semple, of the University of Edinburgh, has been appointed professor of mathematics at Queen's University, Belfast.

Associate Professor Karl Menger, of the University of Vienna, has been appointed visiting lecturer at Harvard University for the first semester and at Rice Institute for the second semester of the present academic year. At Rice Institute he will give an undergraduate course on Metrical Geometry and a graduate course on Theory of Dimension and Curve Theory.

Professor Ella E. Bernstorf, of Friends University, Wichita, has been appointed dean of women at the State Teachers College, California, Pa.

Miss Helen Calkins has been appointed professor of mathematics at the Pennsylvania College for Women, Pittsburgh.

Professor E. U. Condon, of the University of Minnesota, has been appointed associate professor of physics at Princeton.

Dr. H. D. Curtis, director of the Allegheny Observatory, has been appointed director of the new observatory and head of the department of astronomy at the University of Michigan.

Dr. Nat Edmonson, Jr., of Rice Institute, has been appointed assistant professor of mathematics at the Texas Technological College.

Assistant Professor J. D. Eshleman, of the University of Pennsylvania, has been appointed acting professor of mathematics at Heidelberg College, Tiffin, Ohio.

Dr. E. K. Haviland has been appointed assistant professor of mathematics at Lincoln University.

Dr. Lena J. Hawks has been appointed head of the department of mathematics at the Georgia State Womans College.

Mr. E. R. Heineman has been appointed assistant professor of mathematics at the Texas Technological College.

Dr. A. W. Hobbs, professor of applied mathematics at the University of North Carolina, has been appointed dean of the College of Liberal Arts of that University.

Professor Alfred Hume, of the University of Mississippi, has been appointed professor of mathematics at Southwestern University, Memphis.

Miss Myra I. Johnson has been appointed professor of mathematics, science, and philosophy at Southern College, Petersburg, Va.

Mr. George Karelitz, of the Westinghouse Electric and Manufacturing Company, has been appointed associate professor of mechanical engineering at Columbia University.

Assistant Professor E. G. Keller, of the University of Texas, has been appointed a mathematician of the General Electric Company.

Dr. P. W. Ketchum, of the University of Illinois, has been promoted to be associate in mathematics.

Mr. J. H. Kusner has been appointed assistant professor of mathematics at the University of Florida.

Professor Alfred Landé, of the University of Tübingen, has been appointed professor of theoretical physics at Ohio State University.

Assistant Professor H. I. Lane, of the University of South Dakota, has been promoted to an associate professorship of mathematics.

Assistant Professor Flora E. LeStourgeon, of the University of Kentucky, has been promoted to an associate professorship of mathematics.

Mr. C. A. Lovell, of the Drexel Institute, has been appointed a member of the technical staff of the Bell Telephone Laboratories.

Professor H. F. Lusk, of the College of the Pacific, has been appointed professor of engineering mathematics at Sacramento Junior College.

Dr. Dorothy McCoy has been appointed professor of mathematics at Belhaven College.

Dr. Morris Marden has been appointed assistant professor of mathematics in the Extension Division of the University of Wisconsin, at Milwaukee.

Mr. W. L. Porter, of Rice Institute, has been appointed professor of mathematics at the Texas Agricultural and Mechanical College.

Assistant Professor W. C. Risselman, of the University of Pittsburgh, has been appointed professor of mathematics at the Arizona State Teachers College, Flagstaff.

Dr. P. G. Robinson has been promoted to an associate professorship of mathematics at Iowa State College.

Dr. Vladimir Rojansky has been appointed associate professor of physics at Union College, Schenectady.

Associate Professor J. C. Slater, of Harvard University, has been appointed head of the department of physics at the Massachusetts Institute of Technology.

Mr. C. E. Smith has been appointed assistant in the Chabot Observatory.

Associate Professor F. W. Sparks has been promoted to a professorship of mathematics at the Texas Technological College.

Miss Mildred E. Taylor has been appointed professor and head of the department of mathematics at Mary Baldwin College.

Assistant Professor J. M. Thomas, of the University of Pennsylvania, has been appointed assistant professor of mathematics at Duke University.

Dr. L. H. Thomas, of Trinity College, has been appointed associate professor of theoretical physics at Ohio State University.

Dr. E. L. Thompson has been appointed assistant professor of mathematics at the Texas Technological College.

Mr. C. E. Van Horn has been appointed head of the department of mathematics at Fisk University, Nashville.

Dr. Charles Wexler has been appointed head of the department of mathematics at the Arizona State Teachers College.

Dr. G. T. Whyburn has been appointed associate in mathematics at Johns Hopkins University.

Dr. E. E. Witmer has been appointed assistant professor of physics at the University of Pennsylvania.

Dr. Kathryn Wyant has been appointed professor of mathematics at Northeastern Teachers College, Tahlequah, Oklahoma.

Dr. D. M. Yost has been promoted to an assistant professorship of chemistry at the California Institute of Technology.

The following appointments to instructorships are announced:
University of Alabama: Mr. H. S. Thurston;
University of Arkansas: Mr. Paul Cramer;
Lehigh University: Mr. F. S. Beale;
Long Island University: Mr. A. J. Smith;
United States Naval Academy: Mr. Ernest Hawkins.

Professor Robert Fricke, of the Braunschweig Technical School, died July 22, 1930.

Dr. E. B. Knobel, known especially for his editions of the star catalogues of Ptolemy and Ulugh Beg, died July 25, 1930, at the age of eighty-nine.

Professor J. Petzoldt, of the Berlin Technical School, has died at the age of sixty-seven.

Dr. A. Schur, docent at the University of Bonn, has died at the age of thirty-eight.

Mr. William John Greenstreet, for thirty-one years editor of the Mathematical Gazette, died June 28, 1930, at the age of sixty-nine.

Dr. H. H. Turner, Savilian professor of astronomy at Oxford University, died August 20, 1930, at the age of sixty-nine.

Father John George Hagen, S. J., director of the Vatican Observatory, died September 6, 1930, at the age of eighty-three. Father Hagen was formerly director of the astronomical observatory at Georgetown University, and was for many years a member of the American Mathematical Society.

Mr. I. H. Summers, of the engineering department of the General Electric Company, died recently. Mr. Summers was a member of the American Mathematical Society, as nominee of the General Electric Company.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society of which the reports have not yet been published. Papers are numbered serially throughout this volume.* Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

385. Professor B. W. Jones: *A new definition of genus for ternary quadratic forms.*

H. J. S. Smith and L. E. Dickson have defined the genus of a ternary quadratic form in terms of the quadratic character, with respect to the prime factors of the Hessian, of the integers represented by the form and its reciprocal form. In this paper it is shown that associated with every form f are certain arithmetic progressions A having the property that, for every integer n, $f \equiv n \pmod{N}$ is solvable for N arbitrary if and only if n does not occur in a progression of A; that is, all the integers represented by the form are contained in progressions not among A. These progressions, together with the Hessian, are proved to determine the genus of f as defined by H. J. S Smith and by L. E Dickson. (Received September 26, 1930.)

386. Dr. W. J. Trjitzinsky (National Research Fellow): *A study of indefinitely differentiable and quasi-analytic functions.*

Among the various developments presented in this paper are some relating to certain consequences of the generalized Cauchy formulas, existing for Borel monogenic functions. These functions form an important class of quasi-analytic functions of a complex variable. For the real domain, representations are given of quasi-analytic functions which contain as special cases certain representations due to Carleman and de la Vallée-Poussin. In particular, necessary and sufficient conditions are found under which a given set of constants will constitute the set of initial values (at a point) of a function of a given quasi-analytic class, the quasi-analytic classes being defined by the law of decrease of the coefficients of certain series. (Received October 2, 1930.)

387. Dr. J. A. Shohat: *On the classical orthogonal polynomials.*

Starting from the linear second-order differential equation satisfied by the classical orthogonal polynomials (Legendre, Jacobi, Hermite, Laguerre)

* See pp. 1–2 and p. 45 of the January issue.

$\phi_n(x)$, and using Darboux's formula, the author shows that $K_n(x) = \sum_{i=0}^{n} \phi_i^2(x)$ satisfies a linear differential equation of the *first order*. This leads, in a very simple way, to some results, believed new, concerning $K_n(x)$, in particular, $K_n\binom{n}{b}$, also $\phi_n^1\binom{n}{b}$ and $\int_a^x p(x)\phi_n(x)p_{n-1}(x)dx$, $a < x < b$, where (a, b) is the interval of orthogonality, and $p(x)$ is the corresponding characteristic function. (Received September 25, 1930.)

388. Professor Edward Kasner: *Near-collineations.*

A near-collineation of the plane is defined as a point transformation which converts exactly three simply infinite systems of straight lines into straight lines. If four systems exist, all straight lines are preserved and we have necessarily a collineation (see Kasner, this Bulletin, 1903, p. 46). The author studies certain classes of near-collineations in detail. If two of the systems of lines in both planes are pencils, the third system in both planes is also a pencil. The correspondence between the pencils is not homographic. The following problem is solved: find all near-collineations for which the first system is a pencil converted into a homographic pencil, and the second system in the first plane is a pencil. The analogous theory in space is much more complicated. Near-collineations may preserve $2\infty^2$ or $9\infty^2$ straight lines. (Received October 1, 1930.)

389. Mr. Alfred Korzybski: *On structure.*

After giving the necessary definitions, the author discusses the following topics: similarity of structures; relations as a form of multidimensional order; empirical structures, linguistic structures; the structure of mathematics; the structure of the aristotelian, euclidean, and newtonian systems; the structure of the non-aristotelian, non-euclidean, and non-newtonian systems. Consequences and applications are given. (Received October 2, 1930.)

390. Mr. Alfred Korzybski: *A generalized theory of mathematical types.*

This paper discusses the classical theory of types, introduces structural considerations, and presents the new generalized theory of types. Consequences and applications are given. (Received October 2, 1930.)

391. Mr. Alfred Korzybski: *A non-aristotelian system.*

The author considers a certain non-aristotelian system from the point of view of structure and the generalized theory of types. The "is" of identity on different levels is discussed. An outline of the new system as an applied branch of the study of mathematical foundations is given, with consequences and applications. (Received October 2, 1930.)

392. Professor J. Douglas: *Existence of a surface of absolutely least area bounded by a given contour.*

In his solution of the problem of Plateau, of which the manuscript is in the hands of the editors of the Transactions of this Society, the writer develops

certain formulas relating his fundamental functional $A(g)$ to $S(g)$, the area of the harmonic suface determined by the representation $x_i = g_i(\theta)$ of the given contour Γ, any Jordan curve. In a paper just published (Annals of Mathematics, (2), vol. 31 (1930), pp. 457–469) T. Radó has given an independent solution of the Plateau problem under the restriction that Γ be rectifiable, in which he makes use of a certain theorem of Koebe concerning the possibilty of mapping a polyhedral surface conformally on a circular disc. With the aid of this conformal map, and the above mentioned formulas connecting $A(g)$ with $S(g)$, the writer is able to settle very quickly a question left outstanding in both Radó's paper and his own, namely to show that the minimal surface proved to exist has the absolutely least area of any surface bounded by the given contour. In case the Jordan contour is sufficiently crinkly, there is the possibility that this statement may become vacuously true through the circumstance of every area bounded by the given contour being equal to $+\infty$. (Received September 27, 1930.)

393. Professor J. Douglas: *The problem of Plateau for two contours.*

In his paper *A general formulation of the problem of Plateau* (presented to the Society October 26, 1929, abstract in this Bulletin, vol. 36 (1930), p. 50) the author has given methods adequate to solve this problem for any number of contours and any prescribed topology of the minimal surface. In the present paper the work is carried through for two contours (any two non-intersecting Jordan curves) and a minimal surface homeomorphic with a circular ring. The fundamental functional $A(g^{(1)}, g^{(2)})$ involves elliptic functions with periods 2π, $2\,pi$, where p is the logarithm of the ratio of the radii of the ring. A new feature, not present in the one-contour case (topology of a circular disc) is that the existence of the minimal surface depends on whether a certain equation in p has a finite positive real root. For instance, in the classic case of two co-axial circles, radii a and b, distance between centers d (catenoidal surface), this equation is $p^2(a^2+b^2-2ab\cosh p)+d^2\sinh^2 p=0$. The conformal mapping of a plane region bounded by two Jordan curves on a circular ring, including the one-one continuous nature of the map on the boundaries, is contained in this work as a special case. (Received September 27, 1930.)

394. Professor J. Douglas: *The problem of Plateau for a Möbius leaf.*

Here the methods of the paper cited in the previous abstract are applied to solve the problem of the existence of a minimal surface having the topology of a Möbius leaf bounded by a single given contour (Jordan curve). As canonical region on which to represent the minimal surface conformally (with possible singular points) we choose a circular ring, radii 1 and q, where points elliptically inverse with respect to the concentric circle of radius $q^{1/2}$ are regarded as identical. With the use of a two-sided covering surface, the problem amounts to the two-contour case where the two contours are geometrically coincident. The fundamental functional $B(g)$ involves elliptic functions, and the minimal surface depends for its existence on the possibility of a finite positive real root for a certain equation in $p=\log(1/q)$. (Received September 27, 1930.)

395. Professor J. Douglas: *A problem in the topological differential geometry of surfaces.*

According to a formula of Gauss-Bonnet, $\int\int K d\sigma = 4\pi(1-p)$, where K is the Gaussian curvature, $d\sigma$ the element of area of a closed surface of genus p (supposed to have continuous curvature, no sharp edges). If $p=1$ (topology of the torus), then $\int\int K d\sigma = 0$, which shows that both elliptically and hyperbolically curved regions R_e, R_h must present themselves. In the example of the torus neither R_e nor R_h is shrinkable to a point. The presumption would seem to be that the same is true for every surface of genus 1, but no proof is known. The question relates to the possibility of a solution of the Plateau problem with one contour and of the topology of a torus with a hole in it. If $p>1$, then $\int\int K d\sigma < 0$. It follows that the surface cannot be everywhere elliptically curved. Nor can it be everywhere hyperbolically curved, for that is impossible for a closed surface. Thus the same question can be raised in the case of any genus greater than zero. (Received September 27, 1930.)

396. Mr. Hassler Whitney: *A logical expansion in mathematics.*

In a collection of n objects, let $n(A)$ be the number of those with the property A, $n(\overline{A})$ the number not having this property, $n(AB)$ those with both properties A and B, etc. We may write $n(\overline{A}) = n(1-A) = n - n(A)$, $n(\overline{A}\,\overline{B}) = n(1-A)(1-B) = n - n(A) - n(B) + n(AB)$, and in general $n(\overline{A}_1\overline{A}_2 \cdots \overline{A}_k) = n(1-A_1)(1-A_2)\cdots(1-A_k)$. This expansion is useful when the objects with a given property are easier to count than those without. For example, a fundamental formula giving the number of numbers $<x$ not divisible by a set of numbers $p_1, \cdots, p_k$ is exactly this expansion. As another example, in counting the number of ways of coloring a map, we count only those colorings where any pair of adjacent regions are of different colors. If we use the above expansion, and re-group terms, we get the polynomial formula found by Birkhoff. (See the following abstract.) (Received October 4, 1930.)

397. Mr. Hassler Whitney: *A theory of graphs and their coloring.*

The rank of a graph is the number of vertices minus the number of connected pieces. In coloring a graph, any two vertices joined by an edge must be of different colors. Let n_{ij} be the number of sub-graphs of G of rank i which contain j edges. Then the number of ways of coloring G in λ colors is $\sum(-1)^j n_{ij}\lambda^{v-i}$, if G contains v vertices. This result first found (in a different form) by Birkhoff (Annals of Mathematics, vol. 14, p. 42) is proved by a simple logical expansion. (See the preceding abstract.) Recursion formulas are deduced. The numbers n_{ij} for all i, j give a rather complete description of the graph. Suppose a graph G' is in a single piece, and there are no two graphs H_1, H_2 which form G' if they are joined at a single vertex. Then G' is non-separable. Each number n_{ij} for all graphs G is given by a unique polynomial, whose terms are products of numbers of non-separable sub-graphs of different types of G. If we call the linear terms of this polynomial f_{ij}, n_{ij} is given in terms of f_{kl}, $k \leqq i$, $l \leqq j$, by an arithmetical formula. The question of

when a graph is planar is discussed, and results are found. (Received October 4, 1930.)

398. Dr. A. B. Brown: *Critical sets of an arbitrary real analytic function of n variables.*

The results of earlier papers are extended to the case of a function whose critical points may form an arbitrary complex. This condition is satisfied by any real analytic function of n independent variables. Sets of relations are obtained which reduce to those of earlier papers in the cases there considred. The problem has also been solved independently by Morse (in a paper not yet published) by methods, and with results, more in line with the applications, as compared with the geometrical treatment of the present paper. The author owes thanks to Professor Morse for suggesting the investigation of one question which contributed essentially to the results of the paper. (Received October 4, 1930.)

399. Professor Marston Morse: *The critical points of a function of n variables.*

This paper contains among other results the first treatment of the critical points of a real analytic function of n variables without restriction on the nature of the critical points or the function on the boundary. It includes all previous results known to the author as special cases except the results of Whyburn when the critical values are not isolated. It is much broader than the analytic case. The definition of type numbers does not require the critical set to be a complex, or to consist of a finite number of distinct point sets. It is topologically invariant. In the analytic case the definition is reduced to the determination of relative Betti numbers of regions on analytic manifolds bounded by closed analytic manifolds all without singularity. The relations between the type numbers and the appearance and disappearance of non-bounding cycles is taken up. Finally the function is analytically deformed into a nearby function whose critical points are non-degenerate, lie in the neighborhoods of the given critical sets, and have type number sums at least as great as those of the given sets. (Received October 6, 1930.)

400. Dr. Leo Zippin (National Research Fellow): *On a problem concerning the "aleph-null Bein."*

The following problem has been proposed by Karl Menger: Is each point p of a locally connected continuum C, which contains a subcontinuum not locally connected at p, the vertex of an "aleph-null Bein" in C? (An "aleph-null Bein" is a denumerable sequence of arcs converging to a limit arc, the arcs and limit arc having a single point p in common, and in pairs p only.) A locally connected continuum is constructed having as subset the Knaster continuum (which contains no Jordan arc) in which no "aleph-null Bein" exists. It is shown that every point p, as above, is the vertex of an "omega-Bein"; this last, depending on a result due to Mr. Nöbling, was known to him and to Menger. (Received October 6, 1930.)

401. Dr. Lulu Hofmann: *Plane transformations preserving centers of gravity.* II.

This paper is a continuation of one with the same title read before the Society at the New York meeting in February, 1930. To simplify the wording of the theorems, the condition imposed in (a) [(b)] on the transformation [area] is called the g-condition. The main theorem of the present paper then reads as follows. If in (a) a transformation is to obey the g-condition for a complete set of similar areas (that is, all areas of the set are similar and every area similar to one of the set is contained in it), then ϕ and ψ are harmonic, so that, according to the converse of (1) in the earlier paper, the transformation also obeys the g-condition for all circles. In addition, different sets of areas are studied which in (a) determine the same transformations. Such sets are, for example, the set of all ∞^3 squares parallel to a fixed direction, the set of all ∞^4 squares, the set of all areas such that each area has an axis of symmetry and admits a group of rotations of order greater than two. The last theorem in the earlier paper is corrected so as to describe the set of areas last mentioned. (Received October 24, 1930.)

402. Dr. E. J. McShane: *Semi continuity in the calculus of variations, and absolute minima for space problems.*

A theorem is obtained on the existence of the absolute minimum in the isoperimetric problem which includes all of Tonelli's results on the subject. Certain important theorems, including those of Osgood and Lindeberg and several due to Tonelli, are extended to n dimensions with simplification of the proofs. (Received October 25, 1930.)

403. Professor T. R. Hollcroft: *The bitangential curve.*

The bitangential curve of an algebraic surface is the locus of the points of contact of planes which touch the surface in two points. Its order was found by Cayley for a non-singular surface. Basset (*A Treatise on the Geometry of Surfaces*, Cambridge, 1910, pp. 40 and 280) states that he has been unable to determine the reduction in the order of the bitangential curve due to nodal and cuspidal curves of the surface. In the present paper, by combining certain formulas due to Cayley and Zeuthen, there is found the order of the bitangential curve for a surface of given order which has nodal and cuspidal curves of given orders and also multiple points and planes of given orders. (Received October 18, 1930.)

404. Mr. Ainsley H. Diamond: *Quadrilaterals inscribed and circumscribed to a plane cubic.*

The 54 incomplete quadrilaterals circumscribing and inscribing a non-singular cubic may be divided into 9 sets of 6 each, each set corresponding to one of the inflexions. The diagonals of any quadrilateral of a set intersect in the inflexion corresponding to the set. Again the quadrilaterals may be divided into 6 sets of 9 each. The 12 vertices of 3 quadrilaterals appropriately chosen lie by threes on 12 straight lines if chosen from the same set, or by

threes on 4 straight lines if chosen from different sets. The 12 vertices of 3 quadrilaterals of a set which correspond to 3 collinear inflexions from 2 pairs of triangles, each pair being triply in perspective with respect to these inflexions. A nodal cubic is circumscribed and inscribed by 3 quadrilaterals; a cuspidal cubic by none. (Received October 31, 1930.)

405. Dr. J. L. Dorroh: *Some metric properties of descriptive planes.*

A plane satisfying Axioms I–VIII of Velben's *System of axioms for geometry* is called a *descriptive plane. Limit point* is defined in terms of the interiors of triangles. It is shown that a descriptive plane is metric if it contains a countable set of distinct points which has a limit point and that a descriptive plane is in one-to-one continuous correspondence with an everywhere dense subset on the number-plane if it contains a separate segment. Also, it is shown that not every descriptive plane is metric and that not every metric descriptive plane contains a separate segment. (Received October 10, 1930.)

406. Professor Raymond Garver: *Invariantive aspects of a transformation on the Brioschi quintic.*

A number of more or less general transformations on algebraic equations have been devised which lead to transformed equations whose coefficients are invariants (or covariants) of one or more forms. The present paper considers a certain well-known and important transformation on the Brioschi normal quintic, showing that it, or slight modifications of it, can be interpreted as special cases of three such transformations, and discussing the use of this information in the setting up of the transformed equation. (Received October 31, 1930.)

407. Professor E. R. Hedrick: *Theorems associated with Liouville's Theorem for non-analytic functions.*

In a paper presented before this Society on November 29, 1929, the author showed (see this Bulletin, vol. 36, p. 59) that the theorem that the absolute value of a function of a complex variable cannot have a maximum is extensible to the case of functions that are non-analytic, and that the Liouville theorem is also extensible under certain restrictions. In the present paper, the restrictions imposed are removed, and the theorems are presented in very general form, the only condition not present in the usual theorems being that the jacobian shall not vanish. Similar theorems are proved also for three and for n dimensions. (Received October 29, 1930.)

408. Dr. D. H. Lehmer: *A new calculus of numerical functions.*

There are two outstanding theories of numerical functions based on sums of the type $\sum_{n^{f(a)}} \cdot g(b)$, where the summation extends over all solutions (a, b) of $\psi(a, b)=n$. These theories correspond to $\psi(a, b)=a+b$ and ab, and are respectively the calculus of finite differences and the divisor calculus.

In the present paper are discussed the fundamentals of a calculus in which $\psi(a, b) = [a, b]$, the L.C.M. of a and b. The concepts and properties of Dedekind inversion, ideal multiplication (in the sense of Bell), factorable functions ($f(m)f(n) = f(mn)$ for m and n coprime) and r-th divisors have close analogues in this new theory. The fundamental numerical functions such as ϕ, μ, σ, λ, play important roles in this calculus and new identities between them are disclosed. The essential differences between the divisor calculus and the L.C.M. calculus are of such a nature as to render the algebra of the latter very irregular. (Received October 11, 1930.)

409. Dr. Gordon Pall: *Large numbers are sums of four or five values of a quadratic function of x.*

We can suppose that $f(x) = \frac{1}{2}(mx^2 + nx)$, where either m and n are odd and relative prime, or $\frac{1}{2}m$ and $\frac{1}{2}n$ are integers of opposite parities and relative prime; and $m > 0$. Except when $\frac{1}{2}m$ is odd and $\geqq 5$, every sufficiently large integer is a sum of four values of $f(x)$ for integers x; and, if also $m \neq 6$, this is true for $x \geqq 0$. We obtain an explicit determination of all integers which are sums of five, six, or seven values of $f(x)$, for integers x, and a nearly explicit determination of all which are sums of four values. We find all integers which are not sums of four values of $3x^2 + 2jx$, ($j = 1$ or 2), for integers $x \geqq -k$, where k is given and $\geqq 0$. (Received October 31, 1930.)

410. Professor William M. Whyburn: *Critical sets for functions of n real variables.*

A function $f(x_1, \cdots, x_n)$ with continuous first partial derivatives in a closed and connected portion of real n-space is considered. The critical sets (see this Bulletin, vol. 35, p. 703) of f are grouped into four classes on the basis of the number of complementary domains of these sets which have boundary points in the sets. Conditions are given under which the critical sets are identical with the maximal connected subsets of the collection of critical points of f. In case the number of critical sets of f is finite it is shown that the existence of k minimal sets necessitates the existence of at least $k-1$ sets of intermediate type. The classification is consistent with existing classification of isolated critical points. (Received October 27, 1930.)

411. Professor O. H. Rechard: *The expansion problem associated with a class of irregular ordinary differential boundary value problems.*

This paper investigates the convergence of the expansion of an arbitrary vector in terms of the characteristic solutions of a properly restricted homogeneous vector differential system which is ordinary, linear, and of order n. As is usual in problems of this type, the question of the convergence of the expansion in terms of the characteristic solutions of the system is replaced by the formally equivalent question of the convergence of a sequence of complex contour integrals in the plane of the parameter of the system. The irregularity of the boundary value problem considered arises from the fact that

no restrictions are placed upon the exponents of this parameter in the terms of the characteristic equation of the system. Because of the absence of such restrictions it is in general the case that the convergence of the sequence of integrals cannot be shown by the known methods of procedure. It is shown, however, in any case that the sequence is summable in a defined sense to the arbitrarily given vector. The summation is accomplished by the use of an adaptation of the Riesz typical means. (Received October 1, 1930.)

412. Professor V. G. Grove: *The transformation E of nets.*

In this paper the author extends the notion of the transformation of Ribaucour to nets in ordinary space not necessarily the lines of curvature on the sustaining surfaces. Two nets in relation C are said to be E transforms if an only if every point on the line of intersection of corresponding tangent planes is equidistant from the corresponding points of the sustaining surfaces of the two nets. Several of the properties of the transformation of Ribaucour are also properties of the transformation E. In particular, if a surface S is mapped conformally on a surface $\bar{S}$ by a transformation E of nets on these surfaces, these nets are orthogonal nets in relation K. (Received October 3, 1930.)

413. Dr. H. P. Doole: *An expansion problem for a first order differential equation where the coefficient of y is* λx^h, $(h>0)$, $(-a<x<b)$.

The convergence of the expansion of an arbitrary function $f(x)$ in terms of the solutions of $dy/dx+\lambda x^h y=0$, $(h>0)$, with boundary conditions $y(-a)=y(b)$, $(a\neq b)$, is investigated for all values of h greater than zero. When x^h is positive throughout the interval the expansion converges, when x^h is complex divergence is shown, and when x^h changes sign the expansion converges to $f(x)$ plus additional terms involving $f(-a)$ and $f(b)$. (Received October 13, 1930.)

414. Dr. H. P. Doole: *The determination of the limiting value of the integral used in generalization of a certain lemma.*

The evaluation of the contour integral $\operatorname{Lim}_{m=+\infty}(\frac{1}{2}\pi i)\int_{\Gamma_m} e^{-pz}dz/[1-e^{-2z})z^k]$ is extended to the new cases where k is any positive integer, and where p may have any value. The results are peculiarly interesting in that they contain functions of $[p]$, the largest integer algebraically less than p. This lemma is used in an expansion problem in a paper to follow. (Received October 13, 1930.)

415. Dr. Leo Zippin: *Condition that a locally connected continuum contain an Aleph-Null Bein.*

In answer to a question of Professor Menger's, it is here shown that a necessary and sufficient condition that a locally connected continuum C contain an Aleph-Null Bein, in the sense of Menger, is that there exist in C an arc L, and a set of continua $M_1, M_2, \cdots$ of which L is the limit set, such

that $M_i \cdot M_j = 0$, $i, j = 1, 2, 3, \cdots$ and $i \neq j$, and $M_i \cdot L = 0$ for every i. In this case, every point of L is the vertex of an Aleph-Null Bein of C, and a construction is given. (Received October 14, 1930.)

416. Dr. Ralph P. Agnew: *On range of inconsistency of regular transformations, and allied topics.*

The paper considers into what sequences a sequence of a stated character may be carried by regular sequence-to-sequence transformations. Any given bounded divergent sequence may be carried into any given bounded sequence by a transformation which is not only regular but satisfies also further important conditions. Six theorems are given to show what may be expected of specialized real regular transformations when they are applied to divergent sequences; one of these supplements the *Kernsatz* given by Knopp in the Mathematische Zeitschrift of November 1929. It is also shown that a specialized regular transformation can be found which carries a given unbounded sequence into any given sequence; and that if two sequences converge to the same value, a regular transformation can be found which carries the first into the second. An interesting corollary is the following: Corresponding to each divergent sequence $\{s_n\}$ of complex constants and each complex constant σ there is a regular transformation with a triangular matrix (satisfying important conditions in addition to those necessary to ensure regularity) which evaluates $\{s_n\}$ to σ. (Received October 15, 1930.)

417. Dr. A. A. Albert: *On the Wedderburn norm condition for cyclic algebras.*

In the Transactions of this Society (1914) Professor Wedderburn gave a sufficient condition that a cyclic algebra of order n^2 over a non-modular field F be a division algebra. The condition was $\gamma^r \neq \text{norm } (a)$ for any a in the cyclic field $F(i)$ defining the algebra and any integer r less than n. It has never been shown that this condition is necessary except for the case n a prime. For γ algebras A in sixteen units it seems likely that $\gamma^2 = N(a)$ is possible. The author gives a necessary and sufficient condition on γ that this be so and shows that when the condition is satisfied A is the direct product of two generalized quaternion algebras. Necessary and sufficient conditions that A be a division algebra are given. In particular it is proved that for the interesting case where F is the field of all rational numbers the Wedderburn condition is necessary as well as sufficient. (Received October 17, 1930).

418. Professor R. L. Wilder: *A plane, arcwise connected and connected im kleinen set which is not strongly connected im kleinen.*

In a recent paper G. T. Whyburn (Mathematische Annalen vol. 102, pp. 333–334) raised the question as to whether there exists in the plane an arcwise connected and connected im kleinen set which is not arcwise connected im kleinen. In the present paper there is constructed, in the plane, an arcwise connected set which is uniformly connected im kleinen, but which is not

arcwise connected im kleinen at any point, and, indeed, is not strongly connected im kleinen at any point. (Received October 25, 1930.)

419. Dr. Leo Zippin: *Note on locally cyclicly connected continua.*

A locally cyclically connected continuum C is one in which for every point x and every neighborhood U_x there exists a neighborhood V_x such that every two points of V_x belong to a simple closed curve of C in U_x (compare abstract 36–9–329, G. T. Whyburn). It is readily shown that this condition on C is equivalent to the condition that every point of C is an avoidable point; and the equivalence holds for sets which are connected and connected im-kleinen G-subsets of a locally connected continuum. The purpose of the note is to show that every two points of C form the vertex-set of a "c-spindle" in C, i.e., a set of arcs, uncountably infinite, each arc having the vertex-points as endpoints, and having in pairs no other common point. Moreover if each arc minus its endpoints is regarded as an element, then the set of elements is closed, dense in itself, and totally disconnected. (Received Oct. 29, 1930.)

420. Professor G. E. Wahlin: *The rank equation of an algebra.*

This paper shows that some properties derived by Dickson (*Algebra and their Arithmetics*) for the minimal equation of an element of an algebra can be extended to the rank equation of an albegra A over an infinite field if A has a modulus. (Received October 31, 1930.)

421. J. M. Thomas: *Matrices of integers ordering derivatives.*

Riquier has employed a matrix of integers to arrange the partial derivatives of a set of functions in a more or less determinate order. The matrix establishing a given set of order relations is not uniquely determined. Two matrices defining the same order relations will be called equivalent. The present paper studies the equivalence problem. It gives a method of reducing any such matrix to a canonical form having the property that two matrices are equivalent if and only if identical when reduced to canonical form. All transformations which preserve the order relations established by a given matrix are determined in the sense that every such transformation can be expressed as a product of the transformations described in the paper. Some of these can be applied to all matrices, some to only a restricted class. (Received November 1, 1930.)

422. Professor L. M. Graves: *On the problem of Lagrange.*

For calculus of variations problems in the plane, Tonelli (*Calcolo delle Variazioni*, vol. 2, pp. 89, 318, 486, 557) has derived the equation characterizing a minimizing curve under the very weak hypothesis that the function or functions defining that curve shall satisfy a Lipschitz condition. Using the same restriction on the minimizing functions, a multiplier rule for the problem of Lagrange is obtained, and also an analogue of the Weierstrass condition. The theorems of Hildebrandt and Graves on implicit functions and differential equations in general analysis (Transactions of this Society, vol. 29 (1927),

pp. 127, 514, and especially p. 542) are applied to obtain these results. (Received November 3, 1930.)

423. Professor L. M. Graves: *A transformation of the problem of Lagrange.*

The problem of Lagrange in the calculus of variations, in which admissible curves are required to satisfy certain differential equations, can be transformed into one in which these differential equations are replaced by integral equations. For the transformed problem, a multiplier rule is derived in which the multipliers are all constants. When the inverse transformation is applied to this rule, non-constant multipliers appear, and the well-known form of the multiplier rule is obtained. (Received November 3, 1930.)

424. Miss Grace Shover and Professor C. C. MacDuffee: *Ideal multiplication in a rational linear associative algebra.*

In the extension to rational algebras of the theory of ideals by means of matrices with rational integral elements (MacDuffee, Transactions of this Society, vol. 31, p. 71) a satisfactory treatment of ideal multiplication was lacking. This has now been supplied. If the non-singular left ideals $\mathfrak{a}=(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\mathfrak{b}=(\beta_1, \beta_2, \cdots, \beta_n)$, $\mathfrak{ab}$ are represented by the ideal matrices A, B, C respectively, then C is the greatest common right divisor of the matrices $AS(\beta_i)$, $i=1, 2, \cdots, n$, and also of the matrices $B\overline{R}(\alpha_i)$, where $S(\alpha)$, $\overline{R}(\alpha)$ are the second and transposed first matrices of the number α. In the special case of algebraic fields, a basis in canonical form for the ordinary ideal product is thus readily obtained by rational operations. (Received November 3, 1930.)

425. Professor L. W. Griffiths: *Certain universal functions of generalized polygonal numbers.*

The methods and results in this paper are similar to those in the author's paper *A generalization of the Fermat theorem on polygonal numbers* (Annals of Mathematics, (2) vol. 31 (1930), pp. 1–12.) The summands are the value of $g(x)=x+m(x^2-x)/2$ for $x=0, \pm 1, \pm 2, \ldots$. The sum w of the coefficients $\leqq m-2$. There is no conclusion if $w=m-2$ and the first four coefficients are 1, 1, 2, 3 or 1, 1, 2, 4. Otherwise all universal functions are determined. (Received November 4, 1930.)

426. Miss Marguerite Zeigel: *Principal directions for two dimensional surfaces in hyperspace.*

Several generalizations of principal directions on an ordinary surface have been suggested for two dimensional surfaces in hyperspace. Wilson and Moore (Proceedings of the American Academy of Arts and Sciences vol. 52 (1916) p. 350) after discussing the limitations of a generalization proposed by Kommerell have suggested two other possibilities. This paper suggests another generalization, which in some respects seems more analogous to the corresponding idea for ordinary surfaces. The treatment of principal directions for ordi-

nary surfaces consists in projecting the principal curvature vector upon the normal to the surface, and in taking as principal directions those for which this projection is an extreme. It is proposed to employ the same method for a surface in hyperspace. Since in this case there may be three independent normals to the surface, we select the *mean normal vector* as the one on which to project the principal curvature vector. This normal vector, which is a proper invariant (hence independent of the direction of the curve), is a natural choice. In ordery space, even for minimal surfaces, the proposed principal directions reduce to the usual form. (Received November 4, 1930.)

427. Professor P. R. Rider: *On small samples from certain non-normal universes.*

This paper applies a method previously developed (Biometrika, vol. 21, pp. 124–143) to obtain the distributions of the ratio of mean to standard deviation in small samples from a triangular and from a U-shaped universe. The distributions of means of samples from these two universes are also given. The probability corresponding to an interval of three sample standard deviations on each side of the sample mean is discussed. This statistical study was made possible by a grant-in-aid from the National Research Council for the assistance of a computer. (Received November 4, 1930.)

428. Professor F. L. Wren: *A new theory of parametric problems in the calculus of variations.*

The integral to be minimized is

$$I=\int_{t_1}^{t_2}[2f(x^1, x^2, \cdots, x^n; x^{1\prime}, x^{2\prime}, \cdots, x^{n\prime}]^{1/2}dt=\int_{t_1}^{t_2}[2f(x, x')]^{1/2}dt$$

where $f(x, x')$ is positive, homogeneous of degree two in x', and of class C^{iv} in a certain region of $2n$-dimensional points (x, x'). The first and second variations of the integral I are studied and the necessary conditions for a minimum derived therefrom. In studying the second variation it is found desirable to use orthogonal variations, which are those variations η^i which satisfy the relation $\int_{a\beta}\eta^\alpha x^{\beta_1}=0$. Certain properties of the analogues of the Jacobi equations are discussed and a mechanism for determining conjugate points is set up. The Hamilton-Jacobi theory for the integral I is also discussed. (Received November 5, 1930.)

429. Mr. L. J. Paradiso: *Solution of bounded variation of a functional equation.*

In this paper a functional equation of the type $\phi(x)=f(x)+\lambda\int_a^b K(x, s)d\phi(s)$ is studied, where $K(x, s)$ is of uniformly bounded variation in x for each s in (a, b) and continuous in s for each x in (a, b). In order to develop the theory of this functional equation it is necessary to prove some new theorems involving Stieltjes integrals. The method of successive substitution is applied to obtain the solution $\phi(x)$ of this equation. It is shown that a solution $\phi(x)$ exists whenever the absolute value of λ is less than the reciprocal of the total variation of $K(x, s)$ in x in (a, b) and the solution is displayed in the form $\phi(x)=f(x)-\lambda\int_a^b\Phi(x, s; \lambda)df(s)$ where $\Phi(x, s; \lambda)$ is the resolvent kernel of

$K(x, s)$ defined by an infinite series which is absolutely and uniformly convergent for all x and s in the rectangle $a \leqq x \leqq b$, $a \leqq s \leqq b$. To remove this restriction on λ additional independent hypotheses are imposed on $K(x, s)$. The Fredholm theory of this functional equation is then developed and conditions are obtained for the existence of the integrals involved. (Received November 5, 1930.)

430. Dr. J. H. Roberts: *A continuous curve in 3-space.*

In the author's thesis it is shown that if M is a continuous curve in the plane S, then there is a continuous (1, 1) transformation T of S into itself such that the subset of $T(M)$ within any rational rectangle is the sum of a finite number of connected sets. The present paper gives an example which shows that if in the above statement "the plane" and "rectangle" are respectively replaced by "3-dimensional space" and "cube," then the resulting proposition is not true. It is shown that if M is a continuous curve and G is any uncountable collection of distinct spheres in 3-dimensional space S, then there is at least one sphere R (of radius r) of the set G such that for every positive number ϵ there is a subset K of $R-M$, such that $M-K=s_1+s_2+\cdots+s_n$, where $s_i (i \leqq n)$ is connected, and either lies within the sphere concentric with R and of radius $r+\epsilon$, or lies outside the sphere concentric with R and of radius $r-\epsilon$. (Received November 5, 1930.)

431. Professor H. E. Buchanan: *Small oscillations of the neutral helium atom near the equilateral triangle positions.*

This paper discusses the character of small oscillations of the neutral helium atom, from the standpoint of the classical mechanics, when the masses of the negatively charged electrons are finite. Much new information is thus obtained which is lacking when the masses are considered to be infinitesimal. All the results of Crudelli in his memoir published in the Rendiconti, 1926, are obtained here with less algebra and many new theorems are proved. The characteristic exponents for finite values of the masses are, O, O, $\pm KeVi$, $\pm Ke(\alpha \pm i\beta)$, $\pm KeVi$, $\pm KePi$, where e is the charge on each electron. Thus the only possible periodic oscillations are three in number and their periods are inversely proportional to the charge. (Received November 7, 1930.)

432. Professor Gillie A. Larew: *Discontinuous solutions for the anormal case of the Lagrange Problem in the calculus of variations.*

Caratheodory, (Acta Mathematica, vol. 47) has investigated the Problem of Lagrange in the calculus of variations by the method of geodetic equidistance. This treatment does not require the customary hypothesis ruling out the case called by Hahn "anormal." The difficulty has not, however, been eliminated, since, in the anormal case, the extremals lie on a surface in n-space, and solutions, if they exist, must consist of curves with at least one corner. In this paper it is shown that under proper hypotheses there can be constructed in the neighborhood of any extremal with a corner a family of extremals, each with one corner. This family is shown to form a field in which the usual Weierstrass construction is available. (Received November 7, 1930).

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Mémorial des Sciences Mathématiques

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Under the patronage of la Société mathématique de France, l'Académie des Sciences de Paris, and the scientific academies of Servia, Belgium, Bucarest, Poland, Ukraine, Spain, Prague, Rome (dei Lincei), and Sweden. See this BULLETIN, vol. 31, Nos. 5-6, p. 281.

In addition to twenty-five monographs that appeared before 1928, the following have recently been published.

26. B. GAMBIER, *Déformation des surfaces étudiées du point de vue infinitésimal.* 1928.
27. P. APPELL, *Le problème géométrique des déblais et remblais.* 1928.
28. E. COTTON, *Approximations successives et équations différentielles.* 1928.
29. C. GUICHARD, *Les courbes de l'espace à n dimensions.* 1928.
30. L. ZORETTI, *Les principes de la mécanique classique.* 1928.
31. B. GAMBIER, *Applicabilité des surfaces étudiée du point de vue fini.* 1928.
32. CH. RIQUIER, *La méthode des fonctions majorantes et les systèmes d'équations aux dérivées partielles.* 1928.
33. A. BUHL, *Aperçus modernes sur la théorie des groupes continus et finis.* 1928.
34. H. VERGNE, *Ondes liquides de gravité.* 1928.
35. LÉON LECORNU, *Théorie mathématique de l'élasticité.* 1929.
36. PAUL APPELL, *Sur la décomposition d'une fonction méromorphe en éléments simples.* 1929.
37. G. CERF, *Transformations de contact et problème de Pfaff.* 1929.
38. G. VALIRON, *Familles normales et quasi-normales de fonctions méromorphes.* 1929.
39. T. NAGELL, *L'analyse indéterminée de degré supérieur.* 1929.
40. S. LEFSCHETZ, *Géométrie sur les surfaces et les variétés algébriques.* 1929.

Many others have appeared or are in preparation.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

COLUMBIA, MISSOURI, November 28-29, 1930.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, North Hall, Madison, Wisconsin, not later than November 6. All abstracts received by that date will appear in the Bulletin in advance of the meeting. On Saturday morning, Professor C. N. Moore will deliver an address by invitation of the program committee, on *Types of series and types of summability;* and Professor R. E. Langer will deliver an address on *The zeros of exponential sums and of certain related functions.*

UNIVERSITY OF CALIFORNIA AT LOS ANGELES, November 29, 1930.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, Berkeley, California, not later than November 1. All abstracts received by that date will appear in the Bulletin in advance of the meeting. At the request of the program committee, addresses will be delivered by Professor Harald Bohr on *The theory of Dirichlet series,* and by Professor H. F. Blichfeldt on *The method of the geometry of numbers.*

CLEVELAND, OHIO, ANNUAL MEETING, December 29, 1930, to January 1, 1931.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, North Hall, Madison, Wisconsin, not later than December 1. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than November 7. There will be the usual joint session with Section A of the American Association for the Advancement of Science and with the Mathematical Association of America. There will be invited addresses by Professors G. D. Birkhoff, E. T. Bell, and others. A joint session with the American Statistical Society has been arranged for and it is probable that there will be a joint session with the American Physical Society. Professor E. B. Wilson will deliver the eighth Josiah Willard Gibbs Lecture.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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